# PERTURBATION SOLUTIONS AND ASYMPTOTIC SOLUTIONS IN BOUNDARY LAYER THEORY*

by

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## 1. *Introduction*

The treatment of nonsimilar solution of boundary layer equation with zero pressure gradient as perturbations from Blasius solution was outlined by Stewartson [1]. The discrete eigenvalues and eigenfunctions of the perturbation equation were obtained by Libby and Fox [2]. They have applied these perturbation solutions to solve initial value problems with success when the initial profiles do not differ much from a Blasius profile. In their formulation, the tangential velocity u along the surface with x as the arc length and y as the normal coordinate is written as

$$u(x,y) = f_\eta(x,\eta) \sim f_o'(\eta) + f_{1,\eta}(x,\eta) + f_{2,\eta}(x,\eta) + \dots$$

where $f_o$ is the Blasius solution and $\eta$ is the Blasius variable with $\eta = y(2\nu x)^{-\frac{1}{2}}$ and $\nu$ is the kinematic viscosity. It suffices to discuss only the two dimensional incompressible boundary layer, since the generalization to compressible and axially symmetrical cases can always be achieved by the standard transformation of Mangler-Howarth et al. [3] when the appropriate assumptions are accepted. The perturbation solution $f_1(x,\eta)$ satisfies the following differential equation,

$$f_{1,\eta\eta\eta} + f_o f_{1,\eta\eta} + f_o'' f_1 = 2x\,(f_o' f_{1,x\eta} - f_o'' f_{1,x}) \tag{1.1}$$

the initial condition at $x = x_o$,

$$f_{1,\eta}(x_o,\eta) = F(x_o,\eta) - f_o'(\eta) \tag{1.2}$$

and the boundary conditions

$$f_{1,\eta}(x,o) = f_1(x,o) = 0 \tag{1.3}$$

and $$f_{1,\eta}(x,\eta\to\infty) \to 0 \tag{1.4}$$

The function $F(x_o,\eta)$ is related to the initial profile $g(y)$ by the transformation of the variable y to $\eta$, i.e.,

$$u(x_o,y) = g(y) = g\left[\eta(2\nu x_o)^{\frac{1}{2}}\right] = F(x_o,\eta) \tag{1.5}$$

The solution $f_1(x,\eta)$ is written [2] in the form

$$f_{1,\eta}(x,\eta) = \sum_{n=1,2..} A_n (x/x_o)^{-\lambda_n} N_n'(\eta) \tag{1.6}$$

$\lambda_n$ and $N_n$ are the eigenvalues and the normalized eigenfunctions respectively. The coefficients $A_n$ are defined by means of the orthogonal conditions as follows

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$$A_n = \int_0^\infty [(f_0')^4/f_0''] \left\{ \int_0^\eta [F(x_0, \bar{\eta}) - f_0'(\bar{\eta})]\, d\bar{\eta}/f_0' \right\}' (N_n/f_0')'\, d\eta \qquad (1.7)$$

The discrete eigenvalues are obtained when the weaker boundary condition at $\eta \to \infty$, Eq. (1.4), is replaced by the requirement of exponential decay, i.e.,

$$f_{1,\eta}(x, \eta \to \infty) \to 0 \ [\exp(-\eta)] \qquad (1.8)$$

Since the variable x does not appear explicitely in the boundary layer equation with zero pressure gradient, the solution should depend only on the initial profile $g(y)$ and should be independent of the value of $x_0$ which has been used to designate the initial station $x = x_0$. However, the perturbation solution does depend on the choice of $x_0$ since the basic Blasius solution, the initial condition, eqs. (1.3), (1.5), and differential equation (1.1) depend on $x_0$ when $y$ is transformed to $\eta$.

In Ref. 2, the value of $x_0$ is defined by placing the origin at the leading edge or the stagnation point on the body surface. At the suggestion of Ferri, the possibility of improving the perturbation solutions is demonstrated by Fox and Chen [4] by adjusting the value of $x_0$ along the conventional ideals of matching either the displacement thickness, the momentum thickness or the initial shear stress at wall of the Blasius solution with that of the initial profile.

It is the purpose of this paper to find a unique method of defining $x_0$ by the examination of the solution of a simplified boundary layer equation, namely the heat conduction equation. For the unsteady linear heat flow problems, the exact solutions are available. By a shift of time scale by an appropriate amount, $t_0$, the first two terms of the asymptotic expansion are combined to one to form the optimum one term asymptotic representation. This method of determination of the optimum initial station $t_0$ relies on the knowledge of the exact solution. This method, therefore, cannot be applied to determine the optimum initial station $x_0$ for the perturbation solution of boundary layer equation.

Corresponding to the perturbation solution for the boundary layer equation, the perturbation solution or the series solutions are obtained with an arbitrary shift of time scale by the amount $t^*$ with $t^*>0$. Similarly, when the requirement of solutions with exponential decay is imposed, discrete eigenvalues are obtained. The coefficients will be finite if the initial data also possesses the property of exponential decay. The possession of such property for the initial boundary layer profile has been taken for granted.

The first term of the series solution is identified with the first term of the asymptotic solution with the same shift, $t^*$, in time scale. For large time, the series solution agrees with the asymptotic solution. The first two terms of the series degenerate to one when the time shift $t^*$ is so chosen that the initial data is orthogonal to the second eigenfunction and hence the coefficient of the second term vanishes. The value $t^*$ so defined is identical to the value $t_0$ and the leading term is then identical with the optimum one term asymptotic representation.

For boundary layer solutions, the Blasius solution with any value of $x_0$ for the initial station represents the leading term of the asymptotic solution. The perturbation solution which decays faster in x can be considered as the higher order asymptotic solutions. With this interpretation the Blasius solution will be the optimum one term solution if the value $x_0$ is so chosen that the next term i.e., the first term of the perturbation solution vanishes. $x_0$ is therefore defined by the condition that the deviation of the initial profile from the Blasius solution is orthogonal to the first eigenfunction of the perturbation equation.

The advantage of defining $x_0$ in this manner instead of other methods, e.g., matching of displacement or momentum thickness, is confirmed by comparing the corresponding Blasius solutions with the finite difference solutions for a wide variety of initial profiles.

The identification of the perturbation solutions as asymptotic solutions resolves the indeterminancy in the perturbation solution. In a subsequent paper, the higher order asymptotic solutions will be viewed as perturbation solutions so that the appearance of log terms can be predicted and determined directly.

## 2. *Optimum One Term Asymptotic Representation of Linear Flow of Heat*

The equation of linear flow of heat paralled to y-axis is

$$\partial^2 u/\partial y^2 - (1/k)\,(\partial u/\partial t) = 0 \tag{2.1}$$

where u is the temperature and k the diffusivity. The solution of the equation of linear flow of heat in the infinite region, $-\infty<y<\infty$, with the initial condition,

$$u = g(y) \qquad \text{when} \qquad t = 0 \tag{2.2}$$

can be expressed as an integral of the elementary source solution,

$$u\,(y,t) = \tfrac{1}{2}(\pi kt)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(y')\exp\left[-(y-y')^2/(4kt)\right]dy' \tag{2.3}$$

The standard symbols for the error function erf $\xi$ and its derivatives $\Phi_n(\xi) = d^n \mathrm{erf}(\xi)/d\xi^n$ will be introduced. With $\xi$ defined as $(y'-y)/(4kt)^{\frac{1}{2}}$, the solution u(y, t) can be written as

$$\begin{aligned} u(y,t) &= \tfrac{1}{2}\int_{-\infty}^{\infty} g(y')\,\Phi_1(\xi)d\xi \\ &= \tfrac{1}{2}\int_{\xi^*}^{\infty} g(y')\Phi_1(\xi)d\xi + \tfrac{1}{2}\int_{-\infty}^{\xi^*} g(y')\Phi_1(\xi)d\xi \end{aligned} \tag{2.4}$$

where $\xi^* = -y/(4kt)^{\frac{1}{2}}$ is the value of $\xi$ at $y' = 0$.
By means of integration by parts and the method of induction, it can be shown that

$$\begin{aligned} u(y,t) = {} & \tfrac{1}{2}\sum_{n=1}^{N} (4kt)^{-n/2}\,\Phi_n(\xi^*)\left[(n-1)!\right]^{-1}\int_{-\infty}^{\infty} y^{n-1}\,g(\bar{y})d\bar{y} \\ & + \tfrac{1}{2}(4kt)^{-(N+1)/2}\left[(N-1)!\right]^{-1}\left\{\int_0^{\infty} dy'\Phi_{N+1}(\xi)\int_{y'}^{\infty}(\bar{y}-y')^{N-1}g(\bar{y})d\bar{y}\right. \\ & \left. - \int_0^{\infty} dy'\Phi_{N+1}(\xi)\int_{-\infty}^{y'}(\bar{y}-y')^{N-1}g(\bar{y})d\bar{y}\right\} \end{aligned} \tag{2.5}$$

The first N terms represent the asymptotic expansion of u for large t and the last term, representing the difference, is of the order of $t^{-(N+1)/2}$. Of course, this representation is meaningful only when $y^n g(y)$ is integrable from $-\infty$ to $+\infty$. It is therefore necessary that the initial data g(y) should have the property that

$$g(y) = o(|y|^{-\alpha}) \text{ for any } \alpha > o \text{ as } y \to \infty \tag{2.6}$$

Since the initial data g(y) can always be written as the sum of an even function $[g(y) + g(-y)]/2$ and an odd function $[g(y) - g(-y)]/2$ and the solution depends linearly on g, the solution can be split into even and odd solutions which will be examined separately. With kt replaced by $\nu x$ and u interpreted as the x-component of velocity, the even solution can be readily identified as the far wake solution [6] and the odd solution represents the boundary layer solution along a flat plate subjected to the Oseen-Carrier [7] approximation.

For an even initial profile, the asymptotic representation for the solution becomes

$$u(y,t) = u(-y,t) \sim \tfrac{1}{2}(4kt)^{-\frac{1}{2}}\Phi_1(\xi^*)\int_{-\infty}^{\infty} g(\bar{y})d\bar{y}$$

$$+ \tfrac{1}{2}(4kt)^{-3/2}\Phi_3(\xi^*)\int_{-\infty}^{\infty}(\bar{y}^2/2)g(\bar{y})d\bar{y}$$

$$+ \tfrac{1}{2}\sum_{5,7}^{2N+1}(4kt)^{-n/2}\Phi_n(\xi^*)\int_{-\infty}^{\infty}\bar{y}^{n-1}\left[(n-1)!\right]g(\bar{y})d\bar{y}$$

$$+ 0\left[(t)^{-(2N+3)/2}\right] \tag{2.7}$$

$$\sim \tfrac{1}{2}(4kt)^{-\frac{1}{2}}\Phi_1(\xi^*)\int_{-\infty}^{\infty} g(\bar{y})d\bar{y} + 0(t^{-3/2}) \tag{2.8}$$

With $\Phi_1(\xi) = d(\mathrm{erf}\,\xi^*)/d\xi^* = 2(\pi)^{-\frac{1}{2}}\exp[-y^2/(4kt)]$ the first term represents the source solution at y = 0, t = 0, and of the strength $\int_{-\infty}^{\infty} g(\bar{y})d\bar{y}$ which is the integral of the initial profile.

Since the t-derivative of $t^{-\frac{1}{2}}\Phi_1(\xi^*)$ is $\frac{1}{2}t^{-3/2}\,[-1+2(\xi^*)^2]\Phi_1(\xi^*)$, which is $(1/4)t^{-3/2}\,\Phi_3(\xi^*)$, the second term is proportional to the t-derivative of the first term. The second term can be absorbed by the first term with a shift of time scale from t to $\bar{t} = t+t_o$ where

$$t_o = \frac{1}{2k}\int_{-\infty}^{\infty}\bar{y}^2 g(\bar{y})d\bar{y}\Big/\int_{-\infty}^{\infty} g(\bar{y})d\bar{y} \tag{2.9}$$

and eq. (2.7) becomes

$$u \sim \left[\int_{-\infty}^{\infty} g(\bar{y})d\bar{y}\right]\Phi_1(\xi^*)\left[4k(t+t_o)\right]^{-\frac{1}{2}} + 0(t+t_o)^{-5/2} \tag{2.10}$$

where $\bar{\xi}^* = -y/(4k\bar{t})^{\frac{1}{2}} = -y/[4k(t+t_o)]^{\frac{1}{2}}$.

By the comparison of eq. (2.8) and eq. (2.10), the source solution originated at the instant $t = -t_o$ will give a one term asymptotic representation of u with an arror of the order of $t^{-5/2}$ while the same source solution originated at any other instant will give an error of the order of $t^{-3/2}$.

The optimum one term asymptotic representation of an even solution is therefore a source solution with strength equal to the integral of the initial profile located at y = 0 and originated at the instant $t = -t_o$ where $t_o$ defined by eq. (2.9) represents the second moment of the initial profile normalized by the strength of the source.

For odd initial profile, i.e., g(y) = -g(-y), the asymptotic series of eq. (2.5), retains only terms with n equal to even integers, i.e.,

$$u(y,t) = -u(-y,t) \sim \tfrac{1}{2}(4kt)^{-1}\Phi_2(\xi^*)\int_{-\infty}^{\infty} \bar{y}\ g(\bar{y})\ d\bar{y}$$

$$+ \tfrac{1}{2}\ (4kt)^{-2}\Phi_4(\xi^*)\int_{-\infty}^{\infty}\left[\bar{y}^3/3!\right] g(\bar{y})d\bar{y}$$

$$+ \tfrac{1}{2}\sum_{n=6,8}^{2N}(4kt)^{-n/2}\ \Phi_n(\xi^*)\int_{-\infty}^{\infty}\bar{y}^{n-1}\left[(n-1)!\right]^{-1} g(\bar{y})d\bar{y}$$

$$+ 0\ \left[(t)^{-(N+1)}\right] \qquad (2.11)$$

$$\sim \tfrac{1}{2}\ (4kt)^{-1}\Phi_2(\xi^*)\int_{-\infty}^{\infty}\bar{y}g(\bar{y})d\bar{y} + 0\ (t^{-2}) \qquad (2.12)$$

The first term represents a doublet solution located at y = 0 originated at t = 0 and of the strength of the first moment of the initial profile, i.e. $\int_{-\infty}^{\infty} \bar{y}\ g(\bar{y})d\bar{y}$ or $2\int_{\infty}^{\infty} \bar{y}\ g(\bar{y})d\bar{y}$. The second term is again proportional to the first t-derivative of the first term and can be absorbed by the first term with a shift of time scale from t to $\tilde{t} = t + t'_0$, where

$$t'_0 = [1/(3!\,k)]\int_{-\infty}^{\infty}\bar{y}^3\ g(\bar{y})d\bar{y}/\int_{-\infty}^{\infty}\bar{y}\ g(\bar{y})d\bar{y} \qquad (2.13)$$

and eq. (2.12) becomes

$$u(y,t) = -u(-y,t) \sim \tfrac{1}{2}\left[4k(t+t'_0)\right]^{-1}\Phi_2(\tilde{\xi}^*)\int_{-\infty}^{\infty}\bar{y}\ g(\bar{y})d\bar{y} + 0(t^{-3}) \qquad (2.14)$$

where $\tilde{\xi}^* = -y/(4k\tilde{t})^{\frac{1}{2}} = -y/\left[4k(t+t'_0)\right]^{\frac{1}{2}}$. The doublet representation originated at the instant $t = t'_0$ reduce the error from the order of $t^{-2}$ to $t^{-3}$.

The optimum one term asymptotic representation of an odd solution is therefore a doublet solution with strength equal to the first moment of the initial profile located at y = 0 and originated at the instant $t = -t'_0$ where $t'_0$ defined by eq. (2.13) represents the third moment of the initial profile normalized by the strength of the doublet.

The solution for linear flow of heat is divided into even and odd solutions with respect to the space variable. For each solution, the even or the odd, an optimum shift of time scale is defined such that the second term in the asymptotic expansion for large t disappears.

## 3. *Series Solutions of Linear Flow of Heat*

For boundary layer equations with zero pressure gradient, perturbations from the Blasius solution have been obtained in the form of series solutions. These perturbation solutions not only are asymptotically of higher order than the Blasius solution far downstream, i.e., $x \to \infty$ but also remain smaller than the Blasius solution for region near the initial station in order to justify the linearization of boundary layer equation. Since the equation of heat conduction, eq. (2.1), is linear, there is no reason to look for perturbation solution. The counter part is a series solution with the first ferm as the leading term and the remainder of the series as the perturbation solution. To construct the series solution, new variables $\bar{t}$ and $\bar{\xi}$ are introduced with $\bar{t} = t + t^*$ and $\bar{\xi} = y/(4k\bar{t})^{\frac{1}{2}}$. It should be pointed out here that the constant $t^*$ is not yet defined but has to come out positive to make sense for the initial value problem while in the preceeding asymptotic expansion either a positive or negative value of $x_0$ is acceptable.

By means of separation of variables, the solution $u(\bar{\xi}, \bar{t})$ will be represented by weries of product as follows

$$u = \sum_{\lambda} F_{\lambda}(\bar{t})\Phi_{\lambda}(\bar{\xi}) = \sum_{\lambda} A_{\lambda}\bar{t}^{-\lambda/2}\,\Phi_{\lambda}(\bar{\xi}) \tag{3.1}$$

The partial differential equation yields $F_{\lambda}(\bar{t}) = A_{\lambda}\bar{t}^{-\lambda/2}$ and the differential equation for $\Phi_{\lambda}$,

$$\Phi''_{\lambda} + 2\bar{\xi}\Phi'_{\lambda} + 2\lambda\Phi_{\lambda} = 0 \tag{3.2}$$

where (') represents differentiation with respect to $\bar{\xi}$. The boundary conditions for symmetric solution are

$$\Phi'_{\lambda}(0) = 0 \tag{3.3}$$

and $$\Phi_{\lambda}(\infty) \to 0. \tag{3.4}$$

With this weak boundary condition at $\infty$, the eigenvalue $\lambda$ has a continuous spectrum, $\lambda > 0$. To obtain a discrete spectrum for the series solution, it is necessary to impose a stronger condition, [8],

$$\Phi_{\lambda}(\bar{\xi}) \sim o(\bar{\xi}^{-N}) \text{ for any } N > 0. \tag{3.5}$$

With Eq. (3.5) instead of Eq. (3.4), the eigenvalues are the odd integers 1, 3, 5, .... and the eigenfunctions are the odd derivatives of the error function, i.e.,

$$\Phi_n(\bar{\xi}) = d^n(\operatorname{erf}\bar{\xi})/d\bar{\xi}^n = (-1)^{n-1}\left[2/\pi^{\frac{1}{2}}\right]\exp(-\bar{\xi}^2)H_{n-1}(\bar{\xi})$$

where $H_n$ is the Hermite polynomial [9]. The eigenfunction $\Phi_n$ actually fulfills the condition of exponential decay i.e.,

$$\Phi_n \sim 0\left[\bar{\xi}^n\exp(-\bar{\xi}^2)\right]$$

The necessity of imposing a stronger asymptotic behavior to obtain a discrete spectrum is similar to what has happened in the development of the perturbation solutions in boundary layer theory. In the present case the exact solution is available and the stronger asymptotic behavior can be concluded from the exact solution of eq. (2.1) provided that the initial data fulfills the condition of eq. (2.6). With the discrete eigenvalues, the series solution becomes $u(x, t) = \sum_{1,3,5,\ldots} A_n(\bar{t})^{-n/2}\,\Phi_n(\bar{\xi})$ and the constants $A_n$ is related to the initial data by the orthogonal condition with exp $(\bar{\xi}^2)$ as the weighing function,

$$A_n = (t^*)^{n/2}\int_{-\infty}^{\infty} g\left[\bar{\xi}(4kt^*)^{\frac{1}{2}}\right]\exp(\bar{\xi}^2)\Phi_n(\bar{\xi})d\bar{\xi}\Big/\int_{-\infty}^{\infty}\exp(\bar{\xi}^2)\Phi_n^2(\bar{\xi})d\bar{\xi} \tag{3.6}$$

In particular for $n = 1$, $A_1 = \frac{1}{2}(t^*)^{\frac{1}{2}}\int_{-\infty}^{\infty} g\left[\bar{\xi}(4kt^*)^{\frac{1}{2}}\right]d\bar{\xi} = \frac{1}{2}\int_{-\infty}^{\infty} g(y)dy(4k)^{-\frac{1}{2}}$ and the first term which is $\frac{1}{2}(4k\bar{t})^{-\frac{1}{2}}\Phi_1(\bar{\xi})\int_{-\infty}^{\infty} g(y)dy$ represents the basic source solution initiated at the instant $t = -t^*$. For large value of t or $\bar{t}$, the series solution can be identified with the asymptotic solution independent of $t^*$[10]. A direct formal identification can be made by the following observations

$$\lim_{t^*\to 0} (t^*)^{n/2} H_n\left[y/(4kt^*)^{\frac{1}{2}}\right] \to 2^n y^n/(4k)^{n/2}$$

and $$\lim_{t^*\to 0} A_n \to (\tfrac{1}{2})\int_{-\infty}^{\infty} y^{n-1} g(y)dy/\left[(4k)^{n/2}(n-1)!\right].$$

It should be kept in mind that the series solution for the initial value problem is valid only when $t^* > 0$ so that $\bar{t} = t + t^* \geq t^* > 0$. With this identification $t^*$ can be defined by eq. (2.9) for $t_o$ from the study of the asymptotic expansion of the exact solution. It is therefore desirable to find a different definition for $t^*$ which does not rely on the knowledge of the exact solution.

For a given value of $\bar{\xi}$, the series solution can be considered as an asymptotic solution in $\bar{t}$. The first term would be the optimum one term representation if the coefficient of the second term vanishes, i.e.

$$A_3 = 0 \tag{3.7}$$

or $$\int_{-\infty}^{\infty} g\left[\bar{\xi}(4kt^*)^{\frac{1}{2}}\right]\exp(\bar{\xi}^2)\Phi_3(\bar{\xi})d\bar{\xi} = 0 \tag{3.8}$$

The vanish of this integral means that the initial data is orthogonal to $\Phi_3$. For the determination of $t^*$, the variable $\bar{\xi}$ is replaced by y with $\bar{\xi} = y/(4kt^*)^{\frac{1}{2}}$, the identity $\exp(\bar{\xi}^2)\Phi_3(\bar{\xi}) = 2\pi^{-\frac{1}{2}} H_2(\bar{\xi}) = 2\pi^{-\frac{1}{2}}(4\bar{\xi}^2-2)$ is recalled and equation (3.8) becomes $\int_{-\infty}^{\infty} g(y)\left[y^2/(kt_o) - 2\right] dy = 0$ which in turn yields $t^* = \frac{1}{2k}\int_{-\infty}^{\infty} y^2 g(y)dy/\int_{-\infty}^{\infty} g(y)dy$. This definition of $t^*$ is identical to that in eq. (2.9) which was obtained in a different manner from the asymptotic expressions.

Since the first term is orthogonal to $\Phi_3$, eq. (3.8), can be replaced by

$$\int_{-\infty}^{\infty}\left[g(y) - A_1(t^*)^{-\frac{1}{2}}\Phi_1(\bar{\xi})\right]\exp(\bar{\xi}^2)\Phi_3(\bar{\xi})d\bar{\xi} = 0 \tag{3.9}$$

If the first term of the series solution is identified as the basic solution and the remainder is identified as the perturbation solution, the condition for the determination of the unknown shift of origin, $t^*$, is that the coefficient of the first term in the perturbation solution (the second term of the series solution) should vanish. The equivalent condition, eq. (3.9), states that the deviation of the initial profile from the basic solution should be orthogonal to the first eigenfunction of the perturbation solution (the second eigenfunction of the series solution). The basic solution with $t^*$ so determined will yield the optimum one term asymptotic expansion.

The rule can be shown to be valid also for solutions odd in y [10]. This general rule will be applied in the next section to perturbation solutions in boundary layer theory.

### 4. *Optimum Blasius Solution*

In the formulation of perturbation solutions of boundary layer equation with zero pressure gradient and with nonsimilar initial profiles [2], the basic solution is the Blasius solution. With the Blasius solution starts at the origin of the coordinate system, the location of the initial station $x = x_o$ is arbitrary.

The Blasius solution also represents the leading term in asymptotic solution for large x. The perturbation terms which decays as $x^{-\lambda_n}$ with $\lambda_n = 1, 1.887, 2.818, 3.800, \ldots$ can be considered as part of the higher order asymptotic solutions. In the development of perturbation solutions, the expansion parameter is a measure of the deviation of the initial profile from the Blasius profile. Nevertheless, the first term of the first perturbation solution $f^{(1)}_{1,\eta}$ should be identified with the next term of the asymptotic solution for large x due to the following argument. The second perturbation solutions which have homogenous initial and boundary conditions are nontrival due to the nonhomogenous terms $-f_1 f_{1,\eta\eta} + 2x(f_{1,\eta} f_{1,x\eta} + - f_{1,\eta\eta} f_{1,x})$. They behave as $x^{-2\lambda_1}$ or $x^{-2}$ for large x and the second perturbation solution $f_{2,\eta}(x,\eta)$ should behave likewise, therefore $f_{2,\eta}(x,\eta)$ does not contribute any $x^{-1}$ terms. Similar arguments can be extended to high order perturbation solutions.

The possibility of the arbitrariness in $x_o$ can be explained by the fact that the first eigensolution is proportional to the first x-derivative of the Blasius solution [11] and the first eigenvalue is unity, i.e.,

$$f^{(1)}_{1,\eta}(x,\eta) = A_1 \left(\frac{x}{x_o}\right)^{-1} N'(\eta) = A_1 \left(\frac{x}{x_o}\right)^{-1} \eta f''_o \tag{4.1}$$

and $\lambda_1 = 1.$

On the other hand, the second and higher x-derivatives of the Blasius solution do not fulfill the perturbation equation and the boundary conditions and the eigenvalues $\lambda_n$ are not integers for $n \geq 2$.

The identification of the Blasius solution $f_o$ and the first term of the linear perturbation solution $f^{(1)}_1$ with the first two terms of the asymptotic solution and the identification of the first eigenfunction $f^{(1)}_1$ to the x-derivative of the Blasius solution permit the combination of the first two asymptotic terms to one term, the "optimum" Blasius solution. The combination is accomplished by choosing x such that $f^{(1)}_1$ vanishes or the coefficient $A_1$ in eq. (4.1) vanishes. The condition for the determination of $x_o$ is obtained from eq. (1.7),

$$A_1 = \int_0^\infty \left\{\left[f'_o(\eta)\right]^4 / f''_o(\eta)\right\}\left\{\left(\int_0^\eta \left[F(x_o,\eta_o) + - f'_o(\eta_o)\right] d\eta_o\right) / f'_o(\eta)\right\}' \left[N_1(\eta)/f'_o(\eta)\right]' d\eta = 0 \tag{4.2}$$

This is equivalent to the statement that the deviation of the initial profile from the Blasius solution is orthogonal to the first eigensolution $N_1(\eta)$ or $\eta f'_o - f_o$. With the location of the initial station $x_o$ defined by eq. (4.2), the corresponding Blasius solution will be optimum one term solution in the sense that it will differ from the complete solution asymptotically as $x^{-\lambda_2}$ or $x^{-1.887}$ instead of $x^{-1}$. This method of determination of $x_o$ and the optimum one term solution is parallel to that in the preceding two sections for solutions of simple heat conduction equations.

The usefulness of this optimum Blasius solution will be tested by comparison with the exact numerical solutions of the boundary layer equation [12, 13] for several types of initial profiles.

The first type of initial profiles is associated with the boundary layer along a wall which is permeable from x = 0 to x = L, and is impermeable for x > L. The solution for x < L, is similar and may be obtained from Low [14]. The solution for x > L, is nonsimilar. Since the tangential velocity profile at the station x = L, is continuous [15], the initial profile

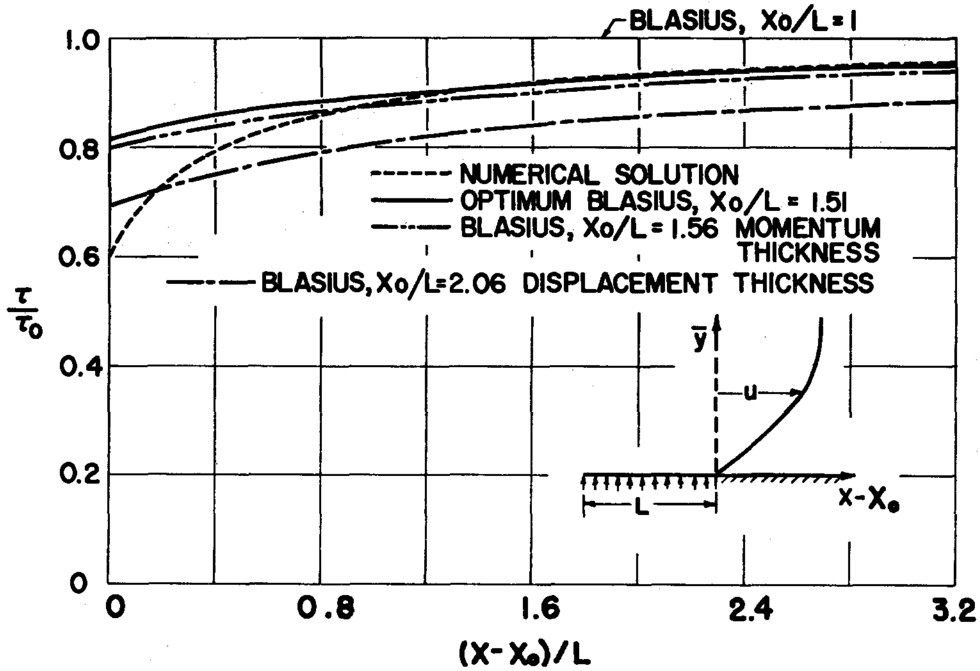


Figure 1 - Variation of the skin friction for after injection, $f_w = -0.5/\sqrt{2}$

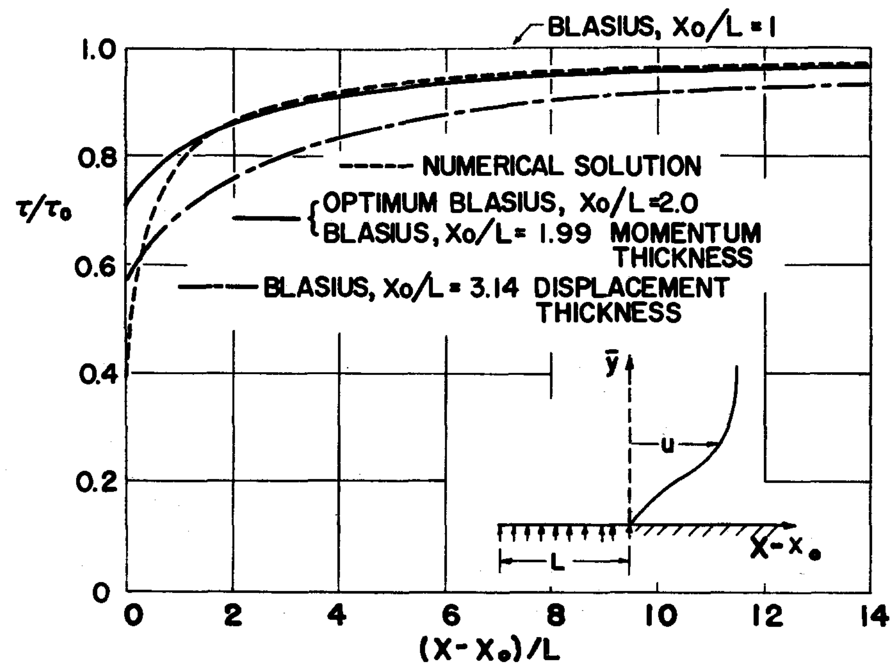


Figure 2 - Variation of the skin friction for after injection, $f_w = -0.75/\sqrt{2}$.

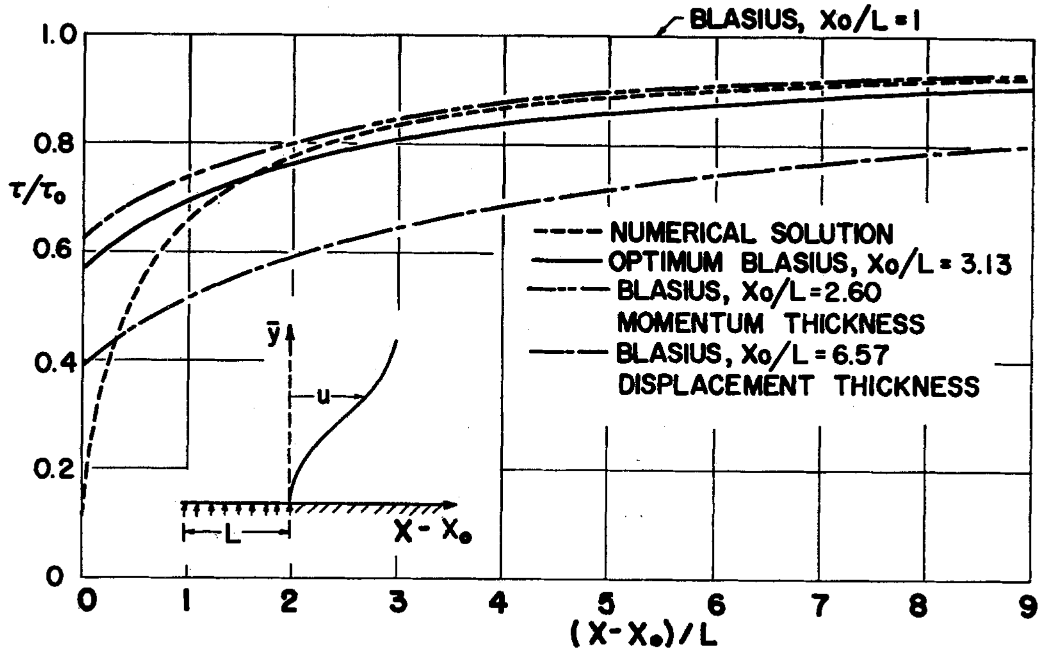


Figure 3 - Variation of the skin friction for after injection, $f_w = -1.0/\sqrt{2}$

at $x = L$ is provided by the continuation of the similar solution from $x < L$ to $x = 1$.

Three initial profiles corresponding to three different rates of injection in the permeable region; $f_w = -0.5(2)^{-\frac{1}{2}}$, $-0.75(2)^{-\frac{1}{2}}$ and $-(2)^{-\frac{1}{2}}$ are considered. Figs. 1, 2 and 3 show the comparison of numerical solutions of shearing stress along the wall with those from Blasius solutions with $x_o$ defined in various manners. The shearing stress is nondimensionalized with respect to the Blasius value originated at $x = 0$. The horizontal line $\tau/\tau_o = 1$ which represents the one term Blasius solution with $x_o = L$ and differs significantly from the numerical solution even at $x - x_o = 10$ L. The Blasius solution with $x_o$ defined by matching the momentum thickness is very close to the optimum Blasius solution with $x_o$ defined by setting $A_1 = 0$. Both solutions are in good agreement with the numerical solution for $x - x_o > 2L$ as shown in Figs. 1, 2, and 3.

All three initial profiles, $g(y)$ are of the type quite similar to Blasius profile in the sense that they are monotonically increasing functions of y and their derivative, $g'(y)$ is monotonically decreasing function of y, except at small value of y where $g'(y)$ increases slightly from $g'(0)$ to a maximum. In the next two examples, the initial profiles will be quite different from the Blasius profile.

Fig. 4 shows an initial profile which simulates the problem of free mixing in the vicinity of a flat wall. The velocity profile at the initial station $x = L$ is composed of a free mixing profile [16] with velocity ratio 0.5014 on top of a Blasius profile. Both the mixing layer and the boundary layer start at $x = 0$. The initial profile $g(y)$ is monotonically increasing in y but the derivative $g'(y)$ decreases to a positive minimum and then increases to a maximum and finally decreases to zero as y increases. The shearing stress at wall is nondimensionalized by the Blasius value originated at $x = 0$ or $x_o = L$. Fig. 4 shows that the Blasius solutions with $x = L$ or with $x_o$ defined by the matching of displacement thickness or momentum thickness are quite different from the numerical solution even for $x - x_o \sim 10$ L. The optimum Blasius solution is approaching the nu-

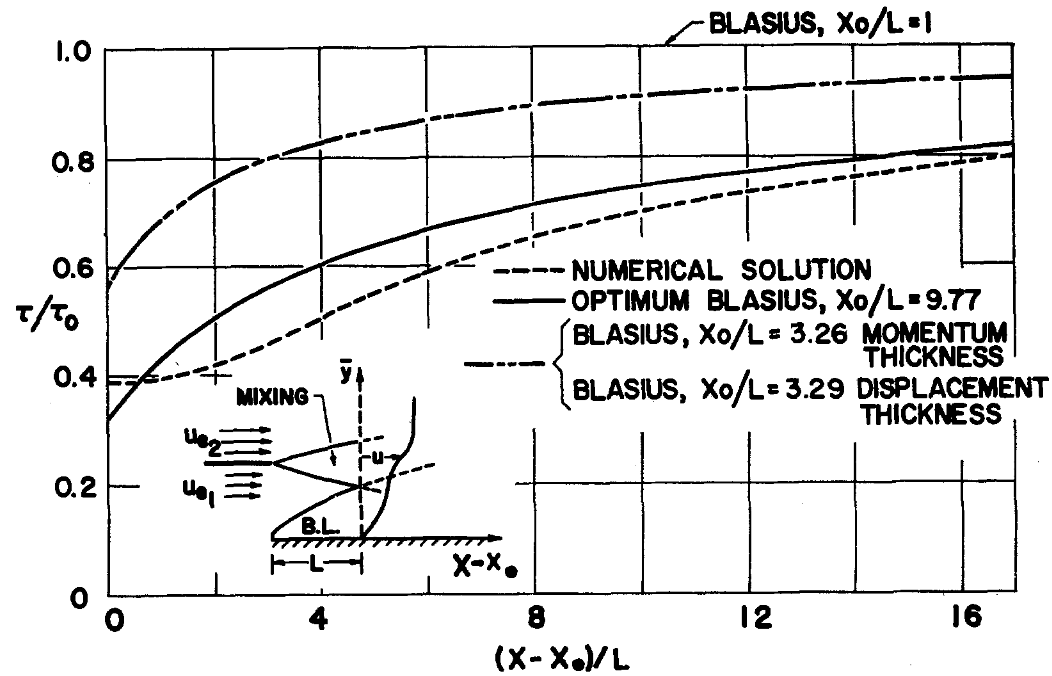


Figure 4 - Variation of the skin friction for mixing boundary layer interaction.

merical solution rapidly for $x - x_o > 2$ L.

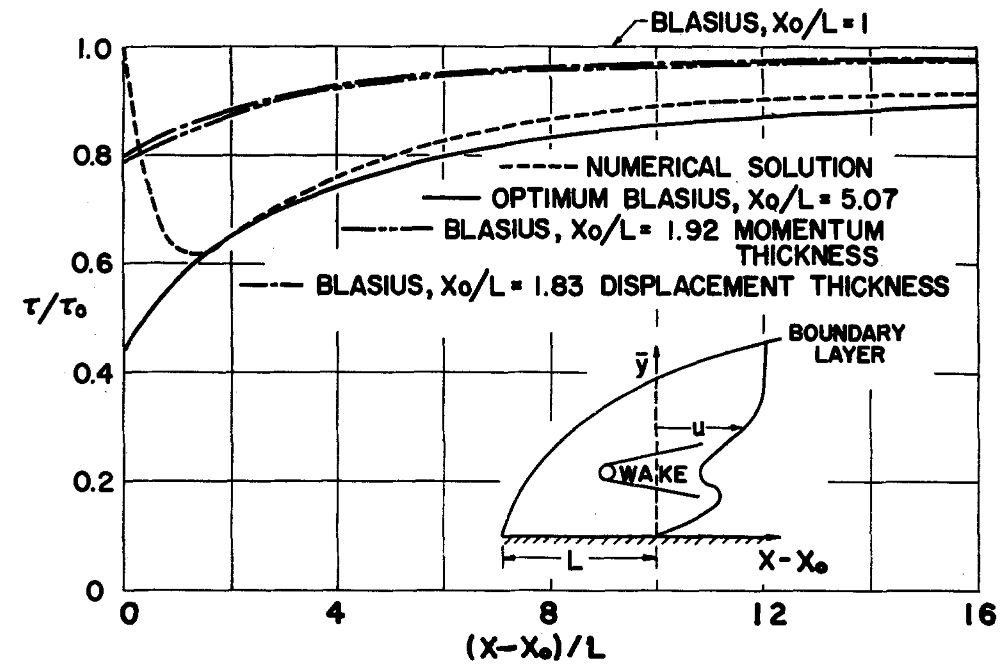


Figure 5 - Variation of skin friction for wake boundary layer interaction.

Fig. 5 shows an initial profile at x = L which simulates the interaction of the wake behind a small cylinder which is submerged inside the boundary layer along the wall originated at x = 0. The engineering objective is to reduce the skin friction or rather the flux at wall. The initial velocity profile is not monotonically increasing with respect to y. There is a reduction in velocity in the wake region. Fig. 5 shows again that the optimum Blasius solution approaches the numerical solution rapidly for $x - x_0 > 2\ L$.

The examples in this section demonstrate that the optimum Blasius solution will approach the exact solution rapidly not only for the cases where the initial profiles deviate slightly from a Blasius profile but also for the cases where the types of profiles are quite different.

## 5. *Concluding Remarks*

The solution of boundary layer equations with zero pressure gradient and a given initial profile by perturbation from Blasius solution depends on the assignment of the distance $x_0$ from the origin of the Blasius solution to the initial station. It is the purpose of this paper to find a formula to determine $x_0$ from a given initial profile with justification. This is accomplished by a systematic study of the relevent properties of the simplified boundary layer equation, the unsteady heat conduction equation.

For unsteady linear heat conduction problems, the asymptotic solutions for large t are obtained from the exact solution of the initial value problem starting at $t = 0$. It is shown that either for an even solution or an odd solution the second term in the asymptotic solution can be absorbed by the first term with a shift of the time scale to form an optimum one term solution. This property is due to the fact that the time derivative of a solution is also a solution of the equation and also preserves the property of even or odd. The even solution is equivalent to the solution of far wake problem and the odd solution is that of the linearized boundary layer equation. The results in Section 2 can be readily extended to two or three dimensional problems.

In Section 3, a series solution for an even or odd linear heat flow problem is developed by a method of separation of variables similar to the variables in perturbation solution of boundary layer equation. There is an arbitrary shift of time scale from t to $\bar{t} = t + t^*$ so that at the initial station $\bar{t} = t^* > 0$. The eigenvalues are discrete subjected to the condition that the solution decays exponentially with respect to the space variable and the series solution exists if the initial profile decays exponentially with respect to the space variable.

The first term of the series solution, the first eigensolution, is identified with the leading asymptotic solution with the same shift of time scale. When the time shift $t^*$ is so chosen that the second term of the series solution vanishes, the first term is identical with the optimum one term asymptotic solution. The corresponding value of $t^*$ is determined by the condition that the second eigensolution is orthogonal to the initial profile or to the deviation of the initial profile from the leading term. Thus $t^*$ and the optimum one term solution are determined without the knowledge of the exact solution. In Section 4 this method is applied to boundary layer problems where the exact solution is not available.

For boundary layer with zero pressure gradient, the Blasius solution can be considered either as the leading term for the perturbation solution or the first term of the asymptotic solution. The development of the perturbation solution with the exception of nonlinear terms is step by step the same as that for the series solution of the heat transfer problems with its leading term identified as the Blasius solution. In particular, the perturbation equation has discrete eigenvalues if the solutions are required to decay exponentially in y and the coefficients of the perturbation series, eq. [1.7] will be finite if the initial profile decays exponentially also. This condition has usually been taken for granted. The first term of the perturbation solution is identifiable with the second term of the asymptotic solution which is proportional to the x-derivative of the Blasius solution. The second term in the asymptotic solution disappears if the first term of the perturbation solution vanishes. The condition for the distance $x_0$ between the origin of the Blasius profile and the initial station is therefore defined by the condition that the deviation of the initial profile from the Blasius solution is orthogonal to the x-derivative of the Blasius solution. The corresponding Blasius solution yields the optimum one term asymptotic solution. The conclusion is also justified by comparison with numerical solutions of several initial profiles whether of the same type or different from the Blasius profile.

It should be noted that instead of integral representations, series solutions with discrete eigenvalue are obtained either from the linear heat transfer problems or for the perturbation problem of boundary layer equations by restricting the initial profiles to the type with exponential decay. This restriction is not severe from the engineering point of view, therefore, the possibility of developing the series solution with a restricted initial profile to other problems should be explored further.

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